THE EXISTENCE OF A DISCONTINUITY IN THE PROFILE OF A SLOW SHOCK MAGNETOHYDRODYNAMIC WAVE

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An analysis is made of the structure of the profile of a slow shock magnetohydrodynamic wave of arbitrary intensity in a nonviscous medium. It is shown that the condition for the formation of a discontinuous profile coincides with the condition for the creation of an isothermal discontinuity in conventional gas dynamics.

1. The system of equations describing the steady-state profile of a plane shock wave in a frame of reference in which the wave is quiescent has the form

$$\sum_{k=1}^{m} \alpha_k a_{ik}(u) \frac{du_k}{dx} = B_i(u) \qquad (i = 1, \ldots, m) \tag{1.1}$$

$$B_i(u) = 0 \qquad (i = m+1, \ldots, n) \tag{1.2}$$

Here $u = \{u\}_1^n$ is a set of parameters describing the state of the medium; $a_{ik}(u)$ and $B_i(u)$ are known finite differential functions of their arguments; α_k are dissipative coefficients such that without loss of generality we can substitute $\alpha_k \neq 0$ $(k = 1, \ldots, m)$, $\alpha_k = 0$ $(k = m+1, \ldots, n)$. The boundary conditions are that when $x \to \pm\infty$ the parameters $u_k(x)$ tend to finite values of $u_k^{\pm}$. These values, obviously, must satisfy the equations

$$B_i(u^{\pm}) = 0 \qquad (i = 1, \ldots, n)$$

Solving Eq. (1.1) with respect to the derivatives, we obtain

$$\alpha_k \frac{du_k}{dx} = \frac{|D_k|}{|a_{ik}|} \qquad (k, i = 1, \ldots, m) \tag{1.3}$$

Equation (1.1) describes the actual dissipative processes in the shock wave, and therefore the determinant $|a_{ik}|$ cannot vanish in the interval (u^-, u^+), i.e., the derivatives du_k/dx $(k = 1, \ldots, m)$ are always finite and the parameters themselves $u_k(x)$ $(k = 1, \ldots, m)$ are continuous.

Differentiating Eqs. (1.2) with respect to x and solving them relative to the derivatives of the remaining n − m parameters, we obtain

$$\frac{du_k}{dx} = \frac{|D_k^*|}{|b_{ik}|}, \qquad b_{ik} = \frac{\partial B_i(u)}{\partial u_k} \qquad (i, k = m+1, \ldots, n) \tag{1.4}$$

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Here $|D_k^*|$ is a determinant consisting of elements of the matrix (b_{ik}) in which the k-column is replaced by a column of functions $C_i(u)$

$$C_i(u) = -\sum_{j=1}^{m} b_{ij}\frac{du_j}{dx} \qquad (i = m+1, \ldots, n) \tag{1.5}$$

The solution of system (1.1) and (1.2) [or system (1.3) and (1.4)] connecting the boundary singular points u^- and u^+ exists, at least in the case of fast and slow shock waves, uniquely [1]. Moreover, if in the interval (u^-, u^+) the determinant $|b_{ik}|$ does not vanish, all the functions $u_k(x)$ also are continuous [2].

Vanishing of the determinant $|b_{ik}|$ in the interval (u^-, u^+) corresponds to transition within the wave profile through the phase velocity of a higher system [ideal system, obtained from Eqs. (1.1) and (1.2) by reversion of the dissipative coefficients α_k to infinity] [2, 3].

It can be seen from Eq. (1.4) that with transition through an n-dimensional surface $|b_{ik}| = 0$ in $(n+1)$-dimensional space of the quantities x, $u_1, \ldots, u_n$, the derivatives du_k/dx $(k = m+1, \ldots, n)$ change sign, passing through infinity, which corresponds either to an unlimited increase or ambiguity of the $n-m$ functions of $u_k(x)$ $(k = m+1, \ldots, n)$ and confirms the absence of a continuous solution with respect to x.

The singular points u^c constitute an exception, in which simultaneously with the determinant $|b_{ik}|$ all determinants $|D_k^*|$ vanish, which, as can be seen easily, occurs when the condition

$$C_{m+1}\lambda_{m+1} + C_{m+2}\lambda_{m+2} + \ldots + C_n\lambda_n = 0$$

is satisfied, where λ_i is the root of the homogeneous system

$$b_{m+1,k}\lambda_{m+1} + b_{m+2,k}\lambda_{m+2} + \ldots + b_{nk}\lambda_n = 0 \qquad (k = m+1, \ldots, n)$$

Thus, the profile of the shock wave contains discontinuities if, in the case of transition through the phase velocity of a higher system (in the future, for brevity we shall call it critical), the system of equations (1.1) and (1.2) do not contain singular points in the interval (u^-, u^+) (for example [3], when $m = 1$), or a singular point exists but the solution does not pass through it [4]. If, further, in the case of transition through the critical velocity the solution passes through a singular point, then the question of discontinuity of the solution must be considered separately: in this case, the solution can be both continuous (as we shall show later) and noncontinuous.

2. Let us consider a plane shock magnetohydrodynamic wave in an ideal gas. In magnetohydrodynamic approximation the system of equations describing the profile of the stationary shock wave has the form [6]

$$\begin{gathered}
\frac{c^2}{4\pi\sigma}\frac{dH_\tau}{dx} = v_n H_\tau - v_\tau H_n - C_1 \qquad (H_n = \text{const}) \\
\eta\frac{dv_\tau}{dx} = jv_\tau - \frac{H_n}{4\pi}H_\tau - C_2 \qquad (\rho v_n \equiv j = \text{const}) \\
\left(\zeta + \frac{4}{3}\eta\right)\frac{dv_n}{dx} = jv_n + p + \frac{H^2}{8\pi} - C_3, \qquad p = t\rho \qquad \left(t \equiv \frac{RT}{\mu}\right) \\
\varkappa\frac{\mu}{R}\frac{dt}{dx} = \frac{\gamma}{\gamma-1}jt - j\frac{v^2}{2} + \frac{H_n H_\tau}{4\pi}v_\tau - \frac{H^2}{8\pi}v_n - pv_n + \frac{C_1}{4\pi}H_\tau + C_2 v_\tau + C_3 v_n - C_4
\end{gathered} \tag{2.1}$$

Here H_τ, v_τ and H_n, v_n are the components of the magnetic field tangential and normal to the wave front and the flow velocity of the gas; p, ρ and t are the pressure, density, and internal energy of 1 g of gas; η and ζ are the first and second viscosity; σ and $\varkappa$ are the electrical and thermal conductivities of the gas; C_1, C_2, C_3, and C_4 are constants determined from the boundary conditions.

It is easy, by means of simple calculations, to verify that the critical velocity, transition through which is possible within the profile of the evolutionary magnetohydrodynamic shock wave, exists only in the case when dissipation due to the viscosity of the medium can be neglected. When $\eta = \zeta = 0$, system (2.1) reduces to a system of two differential equations [2, 4]

$$\beta\frac{dh}{dx} = \varphi(u, h), \qquad \chi\frac{du}{dx} = \frac{f(u, h)}{\psi(u, h)} \tag{2.2}$$

Here,

$$u = \frac{v_n(x) - v_n^-}{v_n^-}, \quad h = \frac{H_\tau(x) - H_\tau^-}{H_\tau^-}, \quad \beta = \frac{c^2}{4\pi\sigma v_n^-}, \quad \chi = \frac{\varkappa\mu(\gamma - 1)}{jR}$$

$$\varphi(u, h) = u + (1 - A_n^{-2}) h + uh \tag{2.3}$$

$$\psi(u, h) = \frac{1}{\gamma M^2} - 1 - 2u - A_\tau^{-2}\left(1 - \frac{h}{2}\right) h \equiv \frac{t_u'}{(v_n^-)^2}$$

$$f(u, h) = (M^{-2} - 1) u - A_\tau^{-2} h - {}^1/_2 (\gamma + 1) u^2 - {}^1/_2 A_\tau^{-2}$$

$$\times [A_n^{-2} + \gamma(1 - A_n^{-2})] h^2 - \gamma A_\tau^{-2} uh \left(1 + \frac{h}{2}\right) + \frac{\chi}{\beta} A_\tau^{-2} (1 + u)$$

$$\times (1 + h) \varphi(u, h) \equiv f_0(u, h) - \frac{\chi}{\beta} \frac{t_h'}{(v_n^-)^2} \varphi(u, h)$$

M is the Mach number, A_n is the normal, and A the tangential Alfven number in the undisturbed medium ahead of the wave. For slow shock waves $A_n < 1$.

The limiting singular points $O(u^- = 0, h^- = 0)$ and $A(u^+, h^+)$ are determined from the equations $\varphi(u, h) = 0$ and $f(u, h) = 0$. When the inequalities

$$\psi(O) < 0 \quad (\gamma M^2 > 1), \qquad \psi(A) > 0 \tag{2.4}$$

are satisfied, this corresponds to transition through the critical velocity (in this case, the isothermal velocity of sound).

In this case, the limiting singular points O and A will be saddle points. The integral curve [5] leaves the point O in the direction $z_2(O)$ and enters at the point A in the direction $z_1(A)$ (limiting case $\chi/\beta \ll 1$)

$$z_2(O) = \frac{\chi}{\beta} \frac{\psi(O)\varphi_u'(O)}{f_u'(O)} + O\left(\frac{\chi^2}{\beta^2}\right) > 0 \tag{2.5}$$

$$z_1(A) = -\frac{f_u'(A)}{f_h'(A)} + \frac{\chi}{\beta}\psi(A) \frac{f_u'(A)\varphi_h'(A) - f_h'(A)\varphi_u'(A)}{f_u'(A) f_h'(A)} + O\left(\frac{\chi^2}{\beta^2}\right) > 0$$

In satisfying the conditions of Eq. (2.4), system (2.2) still has one singular point $C(u^c, h^c)$, the coordinates of which are determined from the equations

$$\psi(u, h) = 0, \qquad f(u, h) = 0$$

The point C represents a node with characteristic directions

$$z_1(C) = -\frac{f_u'(C)}{f_h'(C)} + \frac{\chi}{\beta}\varphi(C) \frac{f_u'(C)\psi_h'(C) - f_h'(C)\psi_u'(C)}{f_u'(C) f_h'(C)} + O\left(\frac{\chi^2}{\beta^2}\right) > 0$$

$$z_2(C) = \frac{\chi}{\beta} \frac{\varphi(C)\psi_u'(C)}{f_u'(C)} + O\left(\frac{\chi^2}{\beta^2}\right) > 0 \tag{2.6}$$

so that in the direction $z_2(C)$ there will be a unique integral curve at the node, which in the future we shall call the separatrix of the node.

3. We shall consider in advance the simpler case of a nonthermally conducting medium. The system of equations describing the profile of a shock wave in this case has the form

$$\beta \frac{dh}{dx} = \varphi(u, h), \qquad f_0(u, h) = 0 \tag{3.1}$$

According to Section 1 this system does not contain a singular point within the interval (OA), and on transition through the critical velocity ($f_{0u}'(u, h) = 0$), which in this case is the velocity of sound, all the parameters of the medium with the exception of the magnetic field undergo a discontinuity. Actually, the functions u(x) are the solution of the system of equations

$$\beta \frac{du}{dx} = -\varphi(u, h) \frac{f_{0h}'(u, h)}{f_{0u}'(u, h)}, \qquad f_0(u, h) = 0$$

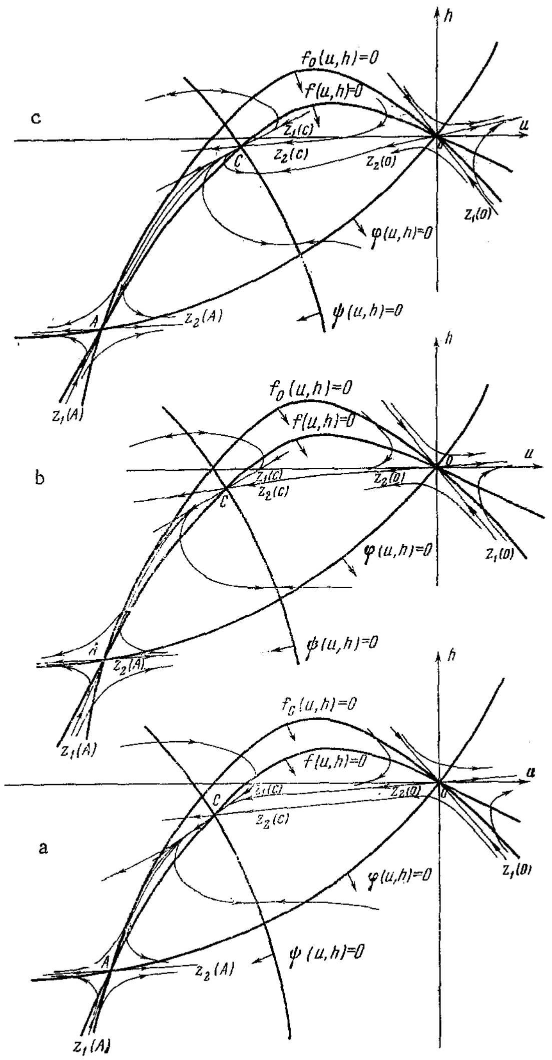

Fig. 1

which can be seen easily when M > 1 contains the nonphysical part du/dx > 0 (rarefaction wave). Hence, it follows immediately that as

$$\left.\frac{du}{dx}\right|_{\substack{u-\\h-}} > 0 \quad \text{for} \quad M > 1 \quad \left(\left.\frac{du}{dx}\right|_{\substack{u+\\h+}} < 0\right)$$

the velocity profile of a supersonic slow shock wave begins with a discontinuity, the magnitude of which is determined from the condition

$$f_0(u,\ 0) = 0 \tag{3.2}$$

Confirmation of the validity of the conclusion drawn can be obtained from expression (2.5), the first part of which tends to zero when $\chi \to 0$, which affects the existence of the part of the wave in which a velocity discontinuity occurs without change of magnetic field.

It is not difficult to verify that Eq. (3.2) gives a value for the velocity discontinuity which coincides in accuracy with the discontinuity in a gas-dynamic shock wave [7]. The same may be said also about the pressure, density, and temperature discontinuities. Thus, a slow supersonic shock magnetohydrodynamic wave in a nonviscous nonthermally conducting medium is initiated by a normal gas-dynamic shock wave [1, 8].

4. Let us return to system (2.2). By eliminating x, we reduced the system of two equations to a single equation relative to dh/du

$$\frac{dh}{du} = \frac{\chi}{\beta}\,\frac{\varphi(u,h)\,\psi(u,h)}{f(u,h)} \tag{4.1}$$

The isoclines

$$\varphi(u,h) = 0, \quad \psi(u,h) = 0 \quad \left(\frac{dh}{du} = 0\right); \qquad f(u,h) = 0 \quad \left(\frac{dh}{du} = \infty\right)$$

and also the known nature of both points (see Section 2) permit the path of the integral curves h(u) [or u(h)] to be satisfiea qualitatively by this equation (see Fig. 1).

The arrows going out from the curves $f = 0$, $\varphi = 0$, and $\psi = 0$ in the figure indicate the region of positive values of the functions listed. The arrows on the integral curves show the direction of motion of the point (u, h) along the integral curve with increase of x [see Eq. (2.2)].

It is easy to see that the integral curve joining the points O and A necessarily passes though point C and that its branch CA has only one common point with the curve $\psi(u,\ h) = 0$. In order to show the validity of the latter statement, we construct the curve $f_0(u,\ h) = 0$ in the plane uh. Because the inequality $f_{0h}'t_u' - f_{0u}'t_h' > 0$ is satisfied at least over the interval OA, the slope of the integral curves at each point on $f_0(u,\ h) = 0$ is less than the slope of the tangent to the curve $f_0(u,\ h) = 0$ at the same point.

On the other hand, on the isocline $f(u,\ h) = 0$, the slope of the integral curves is always greater than the slope of the tangent to this isocline. Hence it follows that in the inverval CA the integral curve is displaced completely to a small region bounded by the curves $f(u,\ h) = 0$ and $f_0(u,\ h) = 0$.

In this case, the solution of Eq. (4.1) has the form [5]

$$u(h) = u_f(h) - \int_{h^+}^{h} \exp\left[\int_s^h F(\alpha)\, d\alpha\right] \frac{du_f(s)}{ds} ds \qquad (h^+ < h < h^c) \tag{4.2}$$

where $u_t(h)$ is a continuous function defined by the relation $f(u, h) \parallel = 0$, and

$$F(h) = \frac{\beta}{\chi} \frac{f_u'(u_f(h), h)}{\psi(u_f(h), h)\, \varphi(u_f(h), h)} \tag{4.3}$$

We note that the function F(h) reverts to infinity at the points $h = h^+$ and $h = h^c$, and within the interval (h^c, h^+) it is negative, but the derivative du_f/dh in this interval is finite and positive. Therefore, the integral in the R.H.S. of Eq. (4.2) does not change sign and vanishes when $h \to h^+$ and $h \to h^c$. Hence it follows that the integral curve on the section CA cannot have any other common point with the curve $\psi(u, h) = 0$ except the point C, and leaves from point C in the direction $z_1(C)$.

Thus, the question of discontinuity of the solution is determined by the behavior of the branch of the integral curve joining the points O and C. Here, the following cases are possible: the integral curve intersects the isocline $\psi(u, h) = 0$ only at the point C and enters it in the direction $z_1(C)$ (Fig. 1a); the integral curve intersects the isocline $\psi(u, h) = 0$ only at point C but enters it in the direction $z_2(C)$ (Fig. 1b) and, finally, before reaching point C the integral curve intersects the isocline $\psi(u, h) = 0$ at one further point (Fig. 1c).

In the first case h(u) is a smooth continuous function, and all parameters of the medium $u_k(x)$ and their derivatives $du_k(x)/dx$ are continuous functions of x.

In the second case the function h(u) has a node at the point C and in this case all the $u_k(x)$ also are continuous but all the derivatives du_k/dx undergo a discontinuity, with the exception of the derivatives of the magnetic field and temperature (weak discontinuity).

In the third case, as can be seen from Fig. 1b, when passing through $\psi(u, h) = 0$ the direction of motion of the point (u, h) along the integral curve, corresponding to an increase of x, changes to the opposite direction. The line $\psi(u, h) = 0$ is singular in the sense that from points of this line located above point C the integral curves diverge and at points located below C they converge from the adjoining regions. Consequently, continuous passage of point O to point C with a monotonic increase of x is impossible. In other words, inside the shock layer all the derivatives du_k/dx and also the function $u_k(x)$ undergo discontinuity with the exception of the magnetic field and the temperature, which remain continuous (isothermic isomagnetic discontinuity).

As shown in Section 3, in the absence of thermal conduction a slow shock magnetohydrodynamic wave is started by a conventional gas-dynamic shock wave. It is obvious that with a conductivity that is nonzero but negligibly small, the condition for the creation of an isothermic discontinuity within the profile of a slow shock wave coincides with the condition for the creation of an isothermic discontinuity in conventional gas-dynamics [7]

$$M^2 > \frac{3\gamma - 1}{\gamma(3 - \gamma)} \tag{4.4}$$

In the case of a finite thermal conductivity, the explanation of the criterion for the creation of a discontinuity in analytical form encounters considerable difficulties; this criterion was estimated by a numerical method.

The most satisfactory method of solving this problem was found to be the determination of the behavior of the separatrix of the node C near the point O. First of all, the solution of Eq. (4.1) for the separatrix has stability in the region $-1 < u < 0$, $-1 < h < 0$, as this is a unique curve leaving the point C in the direction $z_2(C)$ and, secondly, continuous solutions can be distinguished easily from discontinuous solutions. Integration of Eq. (4.1) was undertaken on a computer by the standard method, from point C in the direction of increasing u up to intersection of the separatrix with one of the coordinates of the axis, so that a relative accuracy at the last computed point of not worse than 10^{-5} was guaranteed.

As a result of the calculation, it was found that when

$$M^2 \leqslant \frac{3\gamma - 1}{\gamma(3 - \gamma)} - 0.01$$

independently of the concrete values of the other starting parameters varying over wide limits ($0.2 \leq A_n \leq 0.95$; $0.3 \leq A_\tau \leq 5$, $\chi/\beta = 0.01, 0.1$) the separatrix intersects the axis h below the point O.

In this case the integral curve leaving the point O, as is clear from Fig. 1a, cannot intersect the singular curve $\psi(u, h) = 0$ at any other point except the point C which corresponds to a continuous shock wave profile.

When

$$M^2 \geqslant \frac{3\gamma - 1}{\gamma(3-\gamma)} + 0.01$$

(also independently of the values of the other parameters, see above) the separatrix intersects the axis u to the left of the point O. It can be seen from Fig. 1b that in this case the required integral curve, before reaching the node C, intersects the curve $\psi(u, h) = 0$ at one further point, i.e., the shock wave profile contains an isothermic discontinuity. Thus, within the permissible limits of the calculation carried out, the condition for the creation of an isothermic isomagnetic discontinuity inside the profile of a slow shock magnetohydrodynamic wave coincides with the condition for the creation of an isothermic discontinuity in conventional gas-dynamics.

In conclusion, we note that a limiting case exists in which the profile of a slow shock magnetohydrodynamic wave and, consequently, also the condition for the creation of a discontinuity can be found in analytic form. Actually, in the absence of a tangential component of the magnetic field ($H_\tau^- = 0$) ahead of the wave, system (2.2) acquires the form

$$\beta \frac{dH_\tau}{dx} = (1 - A_n^{-2} + u) H_\tau \tag{4.5}$$

$$\chi \frac{du}{dx} = \left(\frac{f}{\gamma M^2} - 1 - 2u - \frac{H_\tau^2}{8\pi\rho^-(v_n^-)^2}\right)^{-1} \Big\{(M^{-2} - 1)u - \frac{\gamma+1}{2}u^2 - \frac{H_\tau^2}{8\pi\rho^-(v_n^-)^2}\Big[A_n^{-2} + \gamma(1 - A_n^{-2}) + \gamma u - 2\frac{\chi}{\beta}(1+u)(1 - A_n^{-2} + u)\Big]\Big\}$$

It is easy to see that for satisfying the boundary conditions

$$\frac{dH_\tau}{dx} = \frac{du}{dx} = 0 \qquad (x = \pm\infty)$$

a solution is possible only when $H_\tau(x) \equiv 0$ and system (4.5) reduces to one equation

$$\chi \frac{du}{dx} = \left(\frac{1}{\gamma M^2} - 1 - 2u\right)^{-1} \Big\{(M^{-2} - 1)u - \frac{\gamma+1}{2}u^2\Big\}$$

This equation describes the profile of a normal shock wave in a nonviscous gas.

Hence, it follows directly that when $H_\tau^- = 0$ a slow magnetohydrodynamic wave can exist only as a normal gas-dynamic shock wave with the condition for the creation of an isothermal discontinuity Eq. (4.4).

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LITERATURE CITED

1. É. Anderson, Shock Waves in Magnetohydrodynamics [in Russian], Atomizdat, Chap. 6, Moscow (1968).
2. G. Ya. Lyubarskii, The Structure of Shock Waves [in Russian], Prikl. Matem. i Mekhan., 2, No. 6, (1961).
3. G. B. Whitham, Some comments on wave propagation and shock wave structure with application to magnetohydrodynamics, Commun. Pure and Appl. Math., 12, No. 1 (1959).
4. O. I. Dementii and S. V. Dementii, "The structure of fast magnetohydrodynamic shock waves," Magnitnaya Gidrodinam., No. 3 (1968).
5. O. I. Dementii and G. Ya. Lyubarskii, "The theory of low-intensity shock waves," Magnitnaya Gidrodinam., No. 3 (1967).

6. L. D. Landau and E. M. Lifshits, Electrodynamics of Continuous Media [in Russian], Fizmatgiz, Moscow (1959).
7. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], Gostekhizdat, Moscow (1954), Chap. 9, § 85.
8. A. G. Kulikovskii and G. A. Lyubimov, Magnetohydrodynamics [in Russian], Fizmatgiz, Moscow (1962), Chap. 7.